Math 102

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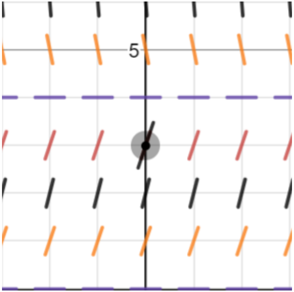
Goals Today - Euler's Method

- An example
- Definition in general
- Practice by hand
- Practice using a spreadsheet
- Conceptual discussion

Euler's Method

$$y' = y(4 - y)$$



Goal: Suppose that $y(0) = 3$. We want to approximate $y(0.5)$.

Euler's Method

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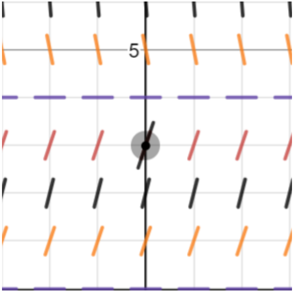

Goal: Suppose that $y(0) = 3$. We want to approximate $y(0.5)$.

Idea: The trajectory must follow the slope field. The slope field gives little local linear approximations to the trajectory. So we can use it to give a step-by-step approximation.

$$y' = y(4 - y) \qquad y(0) = 3$$

Question: What is $y'(0)$?

$$y' = y(4-y) \qquad y(0) = 3$$

Question: What is $y'(0)$?

$$y'(0) = 3(4-3) = 3$$

Therefore,

$$y(0.1) \approx y(0) + 0.1y'(0)$$

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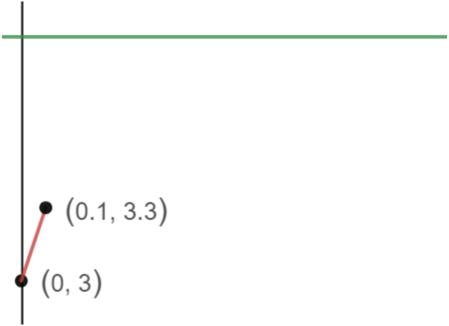
Question: What is $y'(0)$?

$$y'(0) = 3(4-3) = 3$$

Therefore,

$$\begin{aligned} y(0.1) &\approx y(0) + 0.1y'(0) \\ &= 3 + 3(0.1) \\ &= 3.3 \end{aligned}$$



$$y' = y(4-y) \qquad y(0.1) \approx 3.3$$

Question: What is $y'(0.1)$?

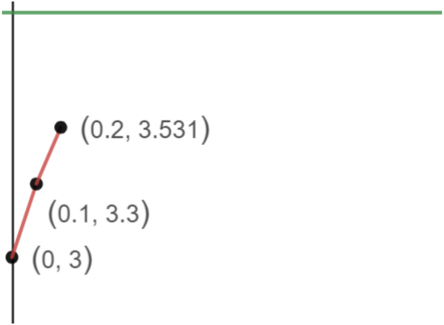
$$y' = y(4-y) \qquad y(0.1) \approx 3.3$$

Question: What is $y'(0.1)$?

$$y'(0.1) = 3.3(4-3.3) = 2.31$$

Therefore,

$$\begin{aligned} y(0.2) &\approx y(0.1) + 0.1y'(0.1) \\ &= 3.3 + 2.31(0.1) \\ &= 3.531 \end{aligned}$$



$$y' = y(4-y) \qquad y(0.2) \approx 3.531$$

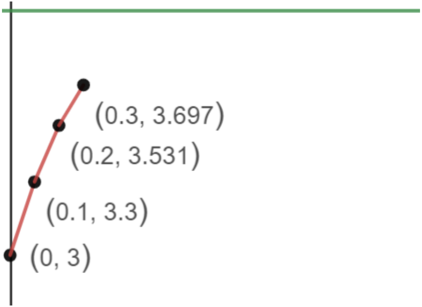
Question: What is $y'(0.2)$?

$$y'(0.2) = 3.531(4-3.531)$$

$$\approx 1.656$$

Therefore,

$$y(0.3) \approx y(0.2) + 0.1y'(0.2)$$
$$\approx 3.697$$



$$y' = y(4-y) \qquad y(0.3) \approx 3.697$$

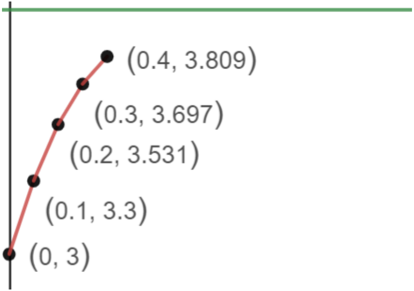
Question: What is $y'(0.3)$?

$$y'(0.3) = 3.697(4-3.697)$$

$$\approx 1.12$$

Therefore,

$$y(0.4) \approx y(0.3) + 0.1y'(0.3)$$
$$\approx 3.809$$



$$y' = y(4-y) \qquad y(0.3) \approx 3.809$$

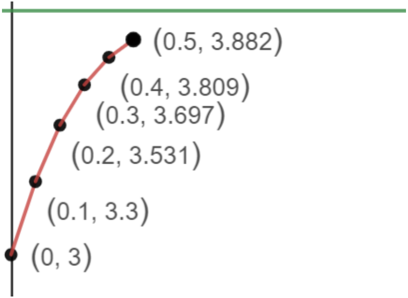
Question: What is $y'(0.4)$?

$$y'(0.4) = 3.809(4 - 3.809)$$

$$\approx 0.728$$

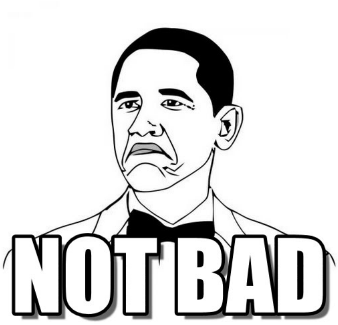
Therefore,

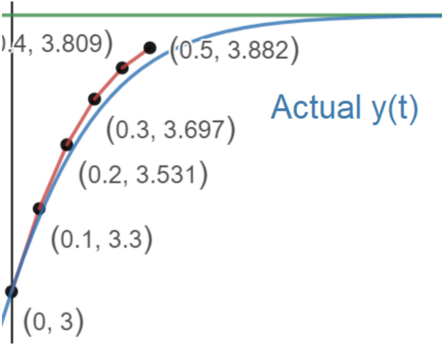
$$y(0.5) \approx y(0.4) + 0.1y'(0.4)$$
$$\approx 3.882$$



$$y' = y(4-y) \qquad y(0.5) \approx 3.882$$

Actual value: $y(0.5) = 3.8273...$





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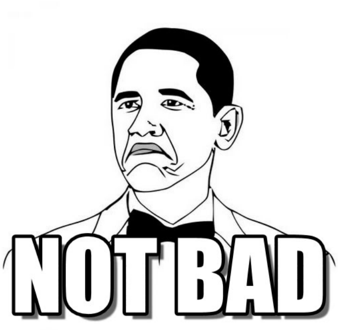
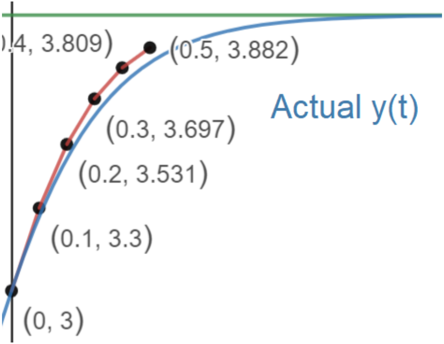
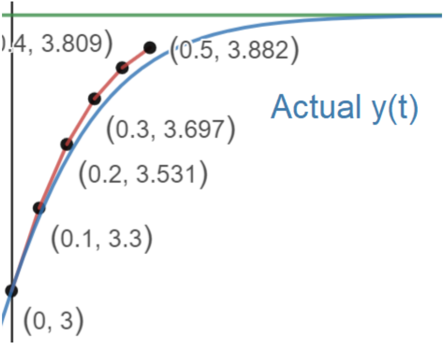




Question: How would we make the approximation better?

$$y' = y(4-y) \qquad y(0.5) \approx 3.882$$

Actual value: $y(0.5) = 3.8273...$



Question: How would we make the approximation better? Use a smaller step size!

Euler's Method

Setup: Suppose that we have *any* differential equation $y' = f(y,t)$ and an initial point (t_0, y_0).

Goal: We want to approximate the value of y at some future time, t_{final}.

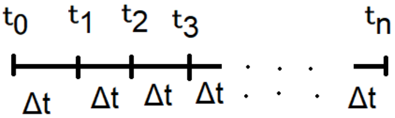
Strategy:

- Choose a step size Δt. Divide the interval $[t_0, t_{\text{final}}]$ into n equal pieces of some size Δt.
- Linear approximation: $(t_0, y_0) \rightsquigarrow (t_1, y_1)$
- Linear approximation: $(t_1, y_1) \rightsquigarrow (t_2, y_2)$
- $\vdots$
- End up with (t_n, y_n).

The smaller Δt is, the better our final approximation will be!

Euler's Method

In more detail: let $\Delta t = \frac{t_{\text{final}} - t_0}{n}$. Let $t_k = t_0 + k\Delta t$.



We have an initial point (t_0, y_0). Iteratively define y_k by

$$y_1 = y_0 + y_0' \cdot (t_1 - t_0) = y_0 + y_0' \cdot \Delta t$$

$$y_2 = y_1 + y_1' \cdot (t_2 - t_1) = y_1 + y_1' \cdot \Delta t$$

$$y_3 = y_2 + y_2' \cdot (t_3 - t_2) = y_2 + y_2' \cdot \Delta t$$

$$\vdots$$

Euler's Method

We are going to use Euler's method with $\Delta t = 0.1$ to approximate $y(0.5)$, given the following

$$y' = y + 2e^{y+1} \qquad y(0.3) = -1$$

Question: How many steps will we have to take?

Question: Write down an expression for the final approximation.

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Two.

Question: Write down an expression for the final approximation.

$$y_0 = -1$$
$$y_1 = -1 + (-1 + 2e^{-1+1})(0.1) = -0.9$$

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Question: How many steps will we have to take?
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Question: Write down an expression for the final approximation.

$$y_0 = -1$$

$$y_1 = -1 + (-1 + 2e^{-1+1})(0.1) = -0.9$$

$$y_2 = -0.9 + (-0.9 + 2e^{-0.9+1})(0.1) = -0.99 + 0.2e^{0.1}$$

Euler's Method

$$y' = 0.1(y-4) \qquad y(0) = 6$$

We want to approximate $y(2)$.

Question: Will Euler's method give us an overapproximation or an underapproximation? (Hint: Draw a picture!)

Euler's Method

$$y' = 0.1(y-4) \qquad y(0) = 6$$

We want to approximate $y(2)$.

Question: Will Euler's method give us an overapproximation or an underapproximation? (Hint: Draw a picture!) Underapproximation. This is because the graph is concave up.

$$y_0'' = 0.1y_0' = 0.01(y_0 - 4) > 0$$

https://www.desmos.com/calculator/qryeddley6

Euler's Method

$$y' = 0.1(y - 4) \qquad y(0) = 6$$

We want to approximate $y(2)$. However, we know the exact answer:

$$y(t) = 4 + Ce^{0.1t}$$

$$y(0) = 6 \implies C = 2$$

So $\boxed{y(2) = 4 + 2e^{0.2}}$.

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We want to approximate $y(2)$. However, we know the exact answer:

$$y(t) = 4 + Ce^{0.1t}$$

$$y(0) = 6 \implies C = 2$$

So $\boxed{y(2) = 4 + 2e^{0.2}}$.

Therefore, using Euler's method **allows us to approximate** $4 + 2e^{0.2}$.

Euler's Method for Approximation

Question: Use a spreadsheet and Euler's method in order to approximate the number e^3. How good of an approximation can you get?

Question: Use a spreadsheet and Euler's method in order to approximate the number $\sqrt{53}$. How good of an approximation can you get?